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Multivariate Spline Functions. II. Best Error Bounds

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The theory of best approximation and best error bounds is sketched. It is observed that spline functions provide the best approximating functions, and then an analysis is presented which obtains best error bounds for the use of two-variable spline functions in interpolation, differentiation, and integration problems.

1. INTRODUCTION

Multivariate spline functions were constructed and discussed in Part I of this paper (Arthur [1]). It was mentioned there that their properties make them useful for interpolation and other practical problems. We construct, in what follows, bounds for these applications.

The type of bound required is that introduced by Golomb and Weinberger [2]. It has the property of being "optimal," in the sense of being attained for some function in the space considered. For the case of interpolation, differentiation and integration in one variable, Secrest [6], [7] has shown the connection with spline functions.

It is our purpose to extend this connection to splines in more than one variable, and we shall require a different method from Secrest's. In Section 2 we quote the results from Part I which we require for our method.

Section 3 contains the theory of best approximation and error bounds, and in Sections 4 and 5 we consider computation of bounds for interpolation, numerical differentiation, and numerical integration. Section 6 contains some concluding remarks.

To simplify the analysis, we shall restrict ourselves to the case of two variables. For more variables, the results are similar, and are obtained by an identical method.

Notation. We shall use, throughout, the notation

$$f_j^i \equiv \frac{\partial^{i+j} f}{\partial x^i \, \partial y^j}. \tag{1.1}$$

2. Two-Variable Spline Functions

Suppose we are given points $\{x_i\}_{i=1}^k$ and $\{y_j\}_{j=1}^l$, where

$$a \leqslant x_1 < x_2 < \cdots < x_k \leqslant b, \qquad c \leqslant y_1 < y_2 < \cdots < y_l \leqslant d, \tag{2.1}$$

and we wish to construct a two-variable spline function, $s(x, y)$, of degree $2m - 1$ in x, and $2n - 1$ in y, satisfying interpolation constraints

$$\lambda_{ij}s = s(x_i, y_j) = r_{ij} \qquad i = 1,..., k, \quad j = 1,..., l. \tag{2.2}$$

Then define

$$K_1(x, s) = \sum_{i=1}^{m} c_i(x)\, c_i(s) + \frac{(-1)^m}{(2m-1)!}\Big[(x-s)_+^{2m-1} + \sum_{i=1}^{m}\sum_{j=1}^{m}(x_i - x_j)_+^{2m-1} \times c_i(x)\, c_j(s) - \sum_{i=1}^{m}\{(x-x_i)_+^{2m-1} c_i(s) + (x_i - s)_+^{2m-1} c_i(x)\}\Big], \tag{2.3}$$

and

$$K_2(y, t) = \sum_{i=1}^{n} d_i(y)\, d_i(t) + \frac{(-1)^n}{(2n-1)!}\Big[(y-t)_+^{2n-1} + \sum_{i=1}^{n}\sum_{j=1}^{n}(y_i - y_j)_+^{2n-1} \times d_i(y)\, d_j(t) - \sum_{i=1}^{n}\{(y-y_i)_+^{2n-1} d_i(t) + (y_i - t)_+^{2n-1} d_i(y)\}\Big], \tag{2.4}$$

where

$$c_i(x) = \prod_{\substack{j=1\\ j\neq i}}^{m} \frac{x - x_j}{x_i - x_j} \qquad i = 1,..., m, \tag{2.5}$$

$$d_i(y) = \prod_{\substack{j=1\\ j\neq i}}^{n} \frac{y - y_j}{y_i - y_j} \qquad i = 1,..., n, \tag{2.6}$$

and

$$z_+ = \begin{cases} z & z \geqslant 0, \\ 0 & z < 0. \end{cases} \tag{2.7}$$

Let

$$K(x, s; y, t) = K_1(x, s)\, K_2(y, t). \tag{2.8}$$

Then

$$s(x, y) = \sum_{i=1}^{k}\sum_{j=1}^{l} \alpha_{ij} K(x, x_i\,; y, y_j), \tag{2.9}$$

where the constants α_{ij} are determined from (2.2).

The properties of this spline function (and its extensions to more general conditions) are discussed in Part I [1].

If λ^* is any linear functional in

$$\mathscr{L}^{(m,n)} = \left\{ \lambda: \lambda f = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \int_a^b \int_c^d f_j^i(x, y)\, d\mu_{ij}(x, y), \right.$$

$$\left. \mu_{ij} \text{ are of bounded variation} \right\}, \tag{2.10}$$

(where f lies in some suitable Hilbert space, e.g. (3.11)), which is independent of those used in (2.2), then we may add another condition,

$$\lambda^* s = r^*, \tag{2.11}$$

to (2.2). The resulting spline s^* is then given by

$$s^*(x, y) = \sum_{i=1}^{k} \sum_{j=1}^{l} \alpha_{ij}^* K(x, x_i\, ; y, y_j) + \alpha^* \lambda^{(s,t)} K(x, s; y, t), \tag{2.12}$$

where the constants are determined from (2.2) and (2.11).

s^* satisfies the important minimization property that

$$\sum_{j=1}^{n} \int_a^b (f_0{}^m(x, y_j))^2\, dx + \sum_{i=1}^{m} \int_c^d (f_n{}^0(x_i\, , y))^2\, dy + \int_a^b \int_c^d (f_n{}^m(x, y))^2\, dy\, dx, \tag{2.13}$$

is at a minimum when $f = s^*$, for all f in a certain Hilbert Space (see (3.11)), satisfying interpolation constraints of the form (2.2) and (2.11).

3. Theory of Best Approximation

As in Golomb and Weinberger [2], and Secrest [6], let H be a real Hilbert Space, and F a continuous linear functional, $F: H \to \mathbb{R}$.

Consider the ball defined by

$$(f, f) \leqslant r^2. \tag{3.1}$$

Suppose we are given $f \in H$ satisfying (3.1) and

$$\lambda_i f = r_i \qquad i = 1,..., n, \tag{3.2}$$

where λ_i are given continuous linear functionals. The problem is: determine $u \in H$ such that

$$\sup_{f \in \mathscr{C}} | F(u) - F(f)| \qquad \text{is minimum,} \tag{3.3}$$

where $\mathscr{C} = \{f \in H: (f, f) \leqslant r^2, \lambda_i f = r_i, i = 1, \ldots, n\}$. We assume that F and $\{\lambda_i\}_{i=1}^n$ are linearly independent, and that $\mathscr{C}$ is nonempty.

Any function $f \in \mathscr{C}$ may be written

$$f = u + \frac{F(f) - F(u)}{F(v)} v + w, \tag{3.4}$$

where v has the properties

$$\lambda_i v = 0, \qquad i = 1, \ldots, n, \qquad \| v \| = 1, \qquad F(v) = \sup_{z \in \mathscr{T}} F(z), \tag{3.5}$$

where

$$\mathscr{T} = \{z: \lambda_i z = 0, i = 1, \ldots, n, \| z \| = 1\}, \tag{3.6}$$

and

$$w \in \{z: \lambda_i z = 0, i = 1, \ldots, n, F(z) = 0\}. \tag{3.7}$$

Now

$$(u, v) = (u, w) = (v, w) = 0, \tag{3.8}$$

and so

$$r^2 \geqslant (f, f) = (u, u) + \left(\frac{F(v) - F(u)}{F(v)}\right)^2 (v, v) + (w, w),$$

or

$$F(u) - F(v)[r^2 - (u, u)]^{1/2} \leqslant F(f) \leqslant F(u) + F(v)[r^2 - (u, u)]^{1/2}. \tag{3.9}$$

$F(u)$ is the *best approximation* to $F(f)$, and has error E bounded by the *best error bound* given by

$$E \leqslant F(v)[r^2 - (u, u)]^{1/2}. \tag{3.10}$$

Now choose H to be the Hilbert Space

$$\{f \in C_{n-1}^{m-1}[a, b; c, d]: f_{n-1}^{m-1} \text{ is absolutely continuous}; \\ f_0^m(x, y_i) \in \mathscr{L}_2[a, b],\ i = 1, \ldots, n; f_n^0(x_i, y) \in \mathscr{L}_2[c, d],\ i = 1, \ldots, m; \\ \text{and } f_n^m \in \mathscr{L}_2[a, b; c, d]\}, \tag{3.11}$$

with scalar product

$$(f, g) = \int_a^b \int_c^d f_n^m(x, y)\, g_n^m(x, y)\, dy\, dx + \sum_{j=1}^n \int_a^b f_0^m(x, y_j)\, g_0^m(x, y_j)\, dx \\ + \sum_{i=1}^m \int_c^d f_n^0(x_i, y)\, g_n^0(x_i, y)\, dy + \sum_{i=1}^m \sum_{j=1}^n f(x_i, y_j)\, g(x_i, y_j), \tag{3.12}$$

where the x_i and y_j are given by (2.1). Let λ_{ij}, $i = 1, \ldots, k$, $j = 1, \ldots, l$ be given by (2.2).

Section 5 of Part I then tells us that the u we require is precisely the spline function given by (2.9).

Now define M by

$$M = \int_a^b \int_c^d (f_n{}^m(x, y))^2 \, dy \, dx + \sum_{j=1}^{n} \int_a^b (f_0{}^m(x, y_j))^2 \, dx + \sum_{i=1}^{m} \int_c^d (f_n{}^0(x_i, y))^2 \, dy. \tag{3.13}$$

Then $r^2 - (u, u)$ becomes

$$U = M - \left[\int_a^b \int_c^d (u_n{}^m)^2 \, dy \, dx + \sum_{j=1}^{n} \int_a^b (u_0{}^m(x, y_j))^2 \, dx + \sum_{i=1}^{m} \int_c^d (u_n{}^0(x_i, y))^2 \, dy \right], \tag{3.14}$$

since u interpolates to f, and our error bound is

$$E \leqslant F(v) \, U^{1/2}. \tag{3.15}$$

By ignoring the part of U containing u, we obtain $E \leqslant F(v) \, M^{1/2}$, which is a bound of Sard-type (see Sard [3, p. 203ff]). Part I tells us that such a bound is "best" in the sense of being attainable. Thus (3.15) is the "best" error bound obtainable for our approximation problem.

In Sections 4 and 5 we shall examine those particular F which give us interpolation, numerical differentiation and numerical integration.

4. Interpolation and Numerical Differentiation

Suppose we choose the functional F to be given by

$$F(f) = f(x^*, y^*). \tag{4.1}$$

If we can identify v, and calculate $v(x^*, y^*)$, then (3.15) will provide an optimal bound for the use of $u(x^*, y^*)$ as an interpolated value from the data (2.2).

Lemma 4.1. *Suppose s is the spline function of degree $2m - 1$ in x and $2n - 1$ in y, satisfying*

$$s(x_i, y_j) = 0, \quad i = 1,..., k, \quad j = 1,..., l, \quad s(x^*, y^*) = 1, \tag{4.2}$$

then s can be expressed uniquely in the form

$$s(x, y) = \alpha^* K(x, x^*; y, y^*) + \sum_{i=1}^{k} \sum_{j=1}^{l} \alpha_{ij} K(x, x_i; y, y_j), \tag{4.3}$$

where α^*, α_{ij} *are constants.*

Proof. This result follows immediately from the theory sketched in Section 2.

THEOREM 4.2. *Let E be the error due to using* $u(x^*, y^*)$ *as an approximation to* $f(x^*, y^*)$. *Then*

$$E \leqslant \frac{U^{1/2}}{(\alpha^*)^{1/2}}, \tag{4.4}$$

where U is given by (3.14), *and* α^* *is the constant in* (4.3).

Proof. If $\| \cdot \|$ is the norm generated by (3.12), and s is as defined in Lemma 4.1, then the minimization property (2.13) shows that

$$v = s / \| s \|, \tag{4.5}$$

has the properties

$$\| v \| = 1, \qquad v(x_i, y_j) = 0, \qquad i = 1,..., k, \quad j = 1,..., l, \tag{4.6}$$

and $v(x^*, y^*)$ is the maximum possible for functions satisfying (4.6). Hence $v(x^*, y^*)$ is the $F(v)$ required for (3.15).

We require $\| s \|$, and so calculate (s, s). The first term is

$$J = \int_a^b \int_c^d (s_n{}^m)^2 \, dy \, dx.$$

Integrate by parts with respect to both variables, use (4.2) and the boundary conditions on s (see [1]), to obtain

$$J = (-1)^{m+n} \int_{x_1}^{x_k} \int_{y_1}^{y_l} s_{2n-1}^{2m-1}(x, y) \, s_1{}^1(x, y) \, dy \, dx$$

$$= (-1)^{m+n} \left(\sum_{i=1}^{k} s(x_i, y^*) \, \beta_{i*} + \sum_{j=1}^{l} s(x^*, y_j) \, \beta_{*j} + \beta_{**} \right), \tag{4.7}$$

where β_{i*} is the coefficient of $(x - x_i)_+^{2m-1} (y - y^*)_+^{2n-1}/(2m-1)! \, (2n-1)!$, β_{*j} is the coefficient of $(x - x^*)_+^{2m-1} (y - y_j)_+^{2n-1}/(2m-1)! \, (2n-1)!$, and β_{**} is the coefficient of $(x - x^*)_+^{2m-1} (y - y^*)_+^{2n-1}/(2m-1)! \, (2n-1)!$ in $s(x, y)$.

Examination of $s(x, y)$ shows that such terms arise only in the function $K(x, x^*; y, y^*)$, and

$$\beta_{**} = (-1)^{m+n} \alpha^*, \tag{4.8}$$

$$\beta_{i*} = \begin{cases} -(-1)^{m+n} \alpha^* c_i(x^*) & 1 \leqslant i \leqslant m, \\ 0 & i > m, \end{cases} \tag{4.9}$$

$$\beta_{*j} = \begin{cases} -(-1)^{m+n} \alpha^* d_j(y^*) & 1 \leqslant j \leqslant n, \\ 0 & j > n. \end{cases} \tag{4.10}$$

The next term in (s, s) is $\sum_{j=1}^{n} \int_a^b (s_0{}^m(x, y_j))^2 \, dx$. Consider

$$\begin{aligned} \int_a^b (s_0{}^m(x, y_1))^2 \, dx &= (-1)^{m-1} \int_a^b s_0^{2m-1}(x, y_1) \, s_0{}^1(x, y_1) \, dx \\ &= (-1)^m \gamma s(x^*, y_1), \end{aligned} \tag{4.11}$$

where γ is the coefficient of $(x - x^*)_+^{2m-1}/(2m-1)!$ in $s(x, y_1)$. Inspection of $K(x, x^*; y, y^*)$ yields

$$\gamma = (-1)^m \alpha^* d_1(y^*). \tag{4.12}$$

Treat the other terms similarly, and also the third part of (s, s) to find

$$\sum_{j=1}^{n} \int_a^b (s_0{}^m(x, y_j))^2 \, dx = \alpha^* \sum_{j=1}^{n} d_j(y^*) \, s(x^*, y_j), \tag{4.13}$$

and

$$\sum_{i=1}^{m} \int_c^d (s_n{}^0(x_i, y))^2 \, dy = \alpha^* \sum_{i=1}^{m} c_i(x^*) \, s(x_i, y^*). \tag{4.14}$$

Now (4.7)–(4.10), (4.13) and (4.14) give

$$(s, s) = \alpha^*. \tag{4.15}$$

U may be computed straightforwardly from the representation (2.9) for u, and then (3.15) gives us the error bound (4.4).

We may treat numerical differentiation similarly, choosing, as the last condition in (4.2), $s_q{}^p(x^*, y^*) = 1$, for estimation of $f_q{}^p(x^*, y^*)$. We use the corresponding linear functional in (2.12), and obtain the same bound (4.4).

5. Numerical Integration

We now choose F to be defined by

$$F(f) = \int_a^b \int_c^d f(x, y) \, dy \, dx, \tag{5.1}$$

and seek a "best" error bound for numerical integration in two variables using spline functions.

LEMMA 5.1. *Suppose s is the spline function of degree $2m - 1$ in x and $2n - 1$ in y, satisfying*

$$s(x_i, y_j) = 0, \quad i = 1, \ldots, k, \quad j = 1, \ldots, l, \qquad \int_a^b \int_c^d s(x, y)\, dy\, dx = 1, \tag{5.2}$$

then s can be expressed uniquely in the form

$$s(x, y) = \alpha^* P(x)\, Q(y) + \sum_{i=1}^{k} \sum_{j=1}^{l} \alpha_{ij} K(x, x_i\, ; y, y_j), \tag{5.3}$$

where

$$P(x)\, Q(y) = \int_a^b \int_c^d K(x, s; y, t)\, dt\, ds, \tag{5.4}$$

and α^*, α_{ij} are constants.

Proof. This follows, as in Lemma 4.1, from the theory sketched in Section 2. Note that the integral in (5.4) can be separated as the product of a function of x and a function of y since K has the same property.

The spline function s provides an extension of the concept of *monospline*, used in the one-variable case by Schoenberg ([4] and [5]).

THEOREM 5.2. *Let E be the error due to using $\int_a^b \int_c^d u(x, y)\, dy\, dx$ as an approximation to $\int_a^b \int_c^d f(x, y)\, dy\, dx$. Then*

$$E \leqslant U^{1/2}/(\alpha^*)^{1/2}, \tag{5.5}$$

where U is given by (3.14), *and α^* is the constant in* (5.3).

Proof. As in Theorem 4.1, we find that $s/\| s \|$ is the function, v, we require. Our bound is obtained after calculation of $F(v) = 1/\| s \|$.

Thus we calculate (s, s). Firstly, note

$$P(x) = \sum_{i=1}^{m} E_i c_i(x) + \frac{(-1)^m}{(2m-1)!} \left[-\frac{(x-a)^{2m}}{2m} + \sum_{i=1}^{m} \sum_{j=1}^{m} E_j (x_i - x_j)_+^{2m-1} c_i(x) - \sum_{i=1}^{m} \left\{ E_i (x - x_i)_+^{2m-1} - \frac{(x_i - a)^{2m}}{2m} c_i(x) \right\} \right], \quad x \in [a, b], \tag{5.6}$$

$$Q(y) = \sum_{j=1}^{n} F_j d_j(y) + \frac{(-1)^n}{(2n-1)!} \left[-\frac{(y-c)^{2n}}{2n} + \sum_{i=1}^{n} \sum_{j=1}^{n} F_j (y_i - y_j)_+^{2n-1} d_i(y) - \sum_{j=1}^{n} \left\{ F_j (y - y_j)_+^{2n-1} - \frac{(y_j - c)^{2n}}{2n} d_j(y) \right\} \right], \quad y \in [c, d], \tag{5.7}$$

where

$$E_i = \int_a^b c_i(t)\,dt, \qquad i = 1,\ldots, m, \tag{5.8}$$

$$F_j = \int_c^d d_j(t)\,dt, \qquad j = 1,\ldots, n. \tag{5.9}$$

The first term of (s, s) is

$$\int_a^b \int_c^d s_n{}^m s_n{}^m \,dy\,dx = (-1)^{m+n} \int_a^b \int_c^d s_{2n}^{2m}(x, y)\, s(x, y)\,dy\,dx. \tag{5.10}$$

(5.3) tells us that s_{2n}^{2m} has the form of a combination of $\delta_{x_i}(x)\,\delta_{y_j}(y)$ plus $\alpha^*(-1)^{m+n}\,[1 + \sum_{i=1}^m E_i \delta_{x_i}(x)][1 + \sum_{j=1}^n F_j \delta_{y_j}(y)]$, where $\delta_{x_i}(x)$ and $\delta_{y_j}(y)$ are Dirac delta functions. (5.2) then shows that (5.10) has the form

$$\alpha^* \int_a^b \int_c^d s(x, y)\,dy\,dx + \alpha^* \sum_{i=1}^m E_i \int_c^d s(x_i, y)\,dy + \alpha^* \sum_{j=1}^n F_j \int_a^b s(x, y_j)\,dx. \tag{5.11}$$

The next term in (s, s) is

$$\sum_{j=1}^n \int_a^b s_0{}^m(x, y_j)\, s_0{}^m(x, y_j)\,dx = (-1)^m \sum_{j=1}^n \int_a^b s_0^{2m}(x, y_j)\, s(x, y_j)\,dx. \tag{5.12}$$

$s_0^{2m}(x, y_j)$ is a linear combination of $\delta_{x_i}(x)$ plus $(-1)^{m+1}\alpha^* F_j$, and so (5.12) becomes

$$-\alpha^* \sum_{j=1}^n F_j \int_a^b s(x, y_j)\,dx, \tag{5.13}$$

and, similarly, the last term in (s, s) is

$$-\alpha^* \sum_{i=1}^m E_i \int_c^d s(x_i, y)\,dy. \tag{5.14}$$

(5.11), (5.13), and (5.14) give

$$\| s \|^2 = (s, s) = \alpha^* \int_a^b \int_c^d s(x, y)\,dy\,dx = \alpha^*, \tag{5.15}$$

and our error bound, (5.5), for numerical integration using spline functions follows from (3.15).

6. Conclusion

The bounds we have obtained in Sections 4 and 5 have the very important property of optimality, in the sense of being attained for some function in the relevant Hilbert Space.

Their computation is, however, extremely time-consuming in practice. Indeed, for interpolation and numerical differentiation, the bounds obtained apply only to one particular point, and not to the whole range. Also, each bound requires more computation than the estimate itself, as another, more complicated, spline function has to be computed, as well as a nontrivial expression, (3.14), involving the given function and the approximating spline function.

Another practical drawback, applying to the method used here, is that computation of spline functions using the $K(x, s; y, t)$ representation tends to be ill-conditioned. [1] gives a stable method suitable for finding the u used in Section 3, but not for finding the s used in Sections 4 and 5.

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